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THERMAL STRESSES AND DEFORMATIONS IN A PLATE SUBJECT TO THE ACTION OF CONCENTRATED ENERGY FLOWS

G. I. Rudin

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A two-dimensional problem is solved concerned with the determination of temperature and stress fields in a plate subject to heating by a radiative flow of Gaussian type.

Nonhomogeneous radiative heating of a plate induces thermal stresses and deformations in the plate. If the intensity of the radiative flow is sufficiently high, the stresses may exceed the limit of strength of the plate material, giving rise to irreversible structural changes in the plate. In particular, the role of the thermal deformations manifests itself in a twisting of the plate surface. If the plate is an element of an optical system, this effect leads to a distortion in the structure of the beam being transmitted, for example, to a lack of focus. There is also increased interest in the study of stresses and deformations under the action of concentrated flows of radiation with a Gaussian distribution of intensity along a radius when the radius of the zone of exposure is equal in order of magnitude or significantly less than the plate thickness. In this case the spatial distribution of stresses and deformations is two-dimensional and differs essentially from the one-dimensional approximation.

In [1] a two-dimensional problem was treated concerned with the determination of the stresses in a free plate under the action of a thermal surface source. At the same time, there is considerable practical interest in the study of stress and deformation fields when the thermal source is a volume source. Such sources are formed, in particular, under the action of a laser beam on a nonmetallic material, and the action of an electron-beam flow on metal. In these cases, a thermal source is formed in the plate whose strength depends expon-

A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 54, No. 3, pp. 477-483, March, 1988. Original article submitted September 29, 1986.

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entially on the longitudinal coordinate (a Burger-type source, [2]). Besides this, we consider in the present paper cases in which one of the plate surfaces is free or is clamped onto a rigid base.

We find the temperature field in a flat plate of thickness d, unbounded in a radial direction, subjected to radiation on it. We assume that the strength of the thermal source, formed as the result of absorption of radiation, can be represented in the form (the flow of radiation is directed on the side of positive z values and is perpendicular to the plane of the plate)

$$W = I_0 A \exp(-\mu z - ar^2) f(t). \tag{1}$$

In the case of absorption of energy from a laser beam, A = μ, the coefficient of absorption, and I_0 is the intensity of the radiation. Under the action of a flow of electrons, the parameters A and μ can be determined from the solution of a problem concerning passage of electrons through the material of the plate [2]. The function f(t) gives the time dependence of the strength of the thermal source. We assume both plate surfaces to be thermally insulated. We determine the temperature field from the solution of the heat conduction equation with appropriate boundary and initial conditions:

$$\begin{gathered}\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{I_0 A}{c\rho}\exp(-\mu z - ar^2) f(t), \\ \left.\frac{\partial T}{\partial z}\right|_{z=0} = \left.\frac{\partial T}{\partial z}\right|_{z=d} = 0, \quad T|_{t=0} = 0.\end{gathered} \tag{2}$$

Applying Hankel and Fourier transformations to the heat conduction equation [3], we obtain

$$\begin{gathered}\frac{d\overline{T}}{dt} = -k\left(p^2 + \frac{n^2\pi^2}{d^2}\right)\overline{T} + \frac{I_0 A \mu d^2}{2c\rho a}\,\frac{1 - \exp(-\mu d)(-1)^n}{(n\pi)^2 + (\mu d)^2} \times \exp\left(-\frac{p^2}{4a}\right) f(t), \\ \overline{T}(p, n, t) = \int_0^d \left[\int_0^\infty r J_0(pr)\, T(r, z, t)\, dr\right] \cos\frac{n\pi z}{d}\, dz,\end{gathered} \tag{3}$$

where n = 0, 1, 2, 3, ... We write the solution of Eq. (3), with the initial condition (2) taken into account, in the form

$$\overline{T} = \frac{I_0 A \mu d^2}{2c\rho a}\exp\left(-\frac{p^2}{4a}\right)\frac{1 - \exp(-\mu d)(-1)^n}{(n\pi)^2 + (\mu d)^2} \times \exp\left[-k\left(p^2 + \frac{n^2\pi^2}{d^2}\right)t\right]\int_0^t f(\theta)\exp\left[k\left(p^2 + \frac{n^2\pi^2}{d^2}\right)\theta\right] d\theta. \tag{4}$$

Carrying out the inverse Fourier and Hankel transformations, we obtain the following relation for the temperature:

$$T(r, z, t) = \frac{I_0 A}{2c\rho a d \mu}\int_0^\infty \exp\left(-\frac{p^2}{4a}\right) p J_0(pr)\left[F_0(t) + \sum_{n=1}^\infty F_n(t) \times \cos\frac{n\pi z}{d}\right] dp, \tag{5}$$

where

$$F_0(t) = [1 - \exp(-\mu d)]\exp(-kp^2 t)\int_0^t f(\theta)\exp(kp^2\theta)\, d\theta,$$

$$F_n(t) = 2\mu^2 d^2 \frac{1 - (-1)^n \exp(-\mu d)}{(n\pi)^2 + (\mu d)^2}\exp\left[-k\left(p^2 + \frac{n^2\pi^2}{d^2}\right)t\right] \times$$

$$\times \int_0^t f(\theta)\exp\left[k\left(p^2 + \frac{n^2\pi^2}{d^2}\right)\theta\right] d\theta, \quad n = 1, 2, 3, \ldots$$

We obtain relations for the displacement components u and w, respectively, in the radial and axial directions [4]:

$$\Delta u - \frac{u}{r^2} + \frac{1}{1 - 2\nu}\frac{\partial \varepsilon}{\partial r} = \frac{2(1 + \nu)}{1 - 2\nu}\,\alpha\,\frac{\partial T}{\partial r}, \tag{6}$$

$$\Delta w+\frac{1}{1-2\nu}\frac{\partial\varepsilon}{\partial z}=\frac{2(1+\nu)}{1-2\nu}\alpha\frac{\partial T}{\partial z}, \qquad (7)$$

where $\varepsilon=\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}$ is the volumetric expansion. We represent u and w in the form

$$u(r,\ z,\ t)=\frac{I_0 A}{2c\rho a d\mu}\int\limits_0^\infty \exp\left(-\frac{p^2}{4a}\right)J_1(pr)\{\varphi_0(z,\ p)F_0(t)+\sum_{n=1}^{\infty}\varphi_n(z,\ p)F_n(t)\}\,dp, \qquad (8)$$

$$w(r,\ z,\ t)=\frac{I_0 A}{2c\rho a d\mu}\int\limits_0^\infty \exp\left(-\frac{p^2}{4a}\right)J_0(pr)\left\{\Phi_0(z,\ p)F_0(t)+\sum_{n=1}^{\infty}\Phi_n(z,\ p)F_n(t)\right\}dp. \qquad (9)$$

Substituting relations (5), (8), and (9) into relations (6) and (7), we obtain equations for the unknown functions φ_n and Φ_n:

$$\frac{d^2\varphi_n}{dz^2}-\frac{p}{1-2\nu}\frac{d\Phi_n}{dz}-\frac{2(1-\nu)}{1-2\nu}p^2\varphi_n=-\frac{2(1+\nu)}{1-2\nu}\alpha p^2\cos\frac{n\pi z}{d}, \qquad (10)$$

$$\frac{2(1-\nu)}{1-2\nu}\frac{d^2\Phi_n}{dz^2}+\frac{p}{1-2\nu}\frac{d\varphi_n}{dz}-p^2\Phi_n=-\frac{2(1+\nu)}{1-2\nu}\alpha p\frac{n\pi}{d}\sin\frac{n\pi z}{d},\quad n=0,\ 1,\ 2,\ \ldots \qquad (11)$$

We now differentiate Eqs. (10) with respect to the variable z and add the result to Eq. (11):

$$\frac{d^2}{dz^2}\left(\frac{d\varphi_n}{dz}+p\Phi_n\right)-p^2\left(\frac{d\varphi_n}{dz}+p\Phi_n\right)=0.$$

We obtain the solution of this equation

$$\frac{d\varphi_n}{dz}+p\Phi_n=A_n\exp(-pz)+B_n\exp(pz), \qquad (12)$$

where A_n and B_n are arbitrary constants to be determined from the boundary conditions. Carrying out elementary transformations on the left side of Eq. (10)

$$(1-2\nu)\left(\frac{d^2\varphi_n}{dz^2}+p\frac{d\Phi_n}{dz}\right)-2(1-\nu)p\frac{d\Phi_n}{dz}-2(1-\nu)p^2\varphi_n=-2(1+\nu)\alpha p^2\cos\frac{n\pi z}{d},$$

we obtain, taking into account the solution (12), the general solution of the nonhomogeneous system of equations (10) and (11):

$$\frac{d\Phi_n}{dz}+p\varphi_n=\alpha p\frac{1+\nu}{1-\nu}\cos\frac{n\pi z}{d}+\frac{1-2\nu}{2(1-\nu)}[B_n\exp(pz)-$$
$$-A_n\exp(-pz)],\ \frac{d\varphi_n}{dz}+p\Phi_n=A_n\exp(-pz)+B_n\exp(pz). \qquad (13)$$

Adding and subtracting equations of the system (13), we obtain two ordinary differential equations of the first order for the functions $(\Phi_n+\varphi_n)$ and $(\Phi_n-\varphi_n)$, whose solution has the form

$$\Phi_n=\alpha pd\frac{1+\nu}{1-\nu}\frac{n\pi\sin\frac{n\pi z}{d}}{(n\pi)^2+(pd)^2}+\frac{A_n}{4(1-\nu)}\left[z+\frac{3-4\nu}{2p}\right]\exp(-pz)-$$
$$-\frac{B_n}{4(1-\nu)}\left[z-\frac{3-4\nu}{2p}\right]\exp(pz)+\frac{C_n}{2}\exp(-pz)+\frac{D_n}{2}\exp(pz), \qquad (14)$$
$$\varphi_n=\alpha p^2d^2\frac{1+\nu}{1-\nu}\frac{\cos\frac{n\pi z}{d}}{(n\pi)^2+(pd)^2}+\frac{A_n}{4(1-\nu)}\left[z-\frac{3-4\nu}{2p}\right]\exp(-pz)+$$
$$+\frac{B_n}{4(1-\nu)}\left[z+\frac{3-4\nu}{2p}\right]\exp(pz)+\frac{C_n}{2}\exp(-pz)-\frac{D_n}{2}\exp(pz).$$

We obtain the arbitrary constants A_n, B_n, C_n, D_n from the boundary conditions for the stresses and displacements. We consider the following cases.

1. Surface z = 0 is free, surface z = d is clamped onto a rigid base. The stress tensor components for z = 0 are

$$\sigma_{rz} = 2G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) = 0,\ \sigma_{zz} = 2G\left[\frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu} \times \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right) - \frac{1+\nu}{1-2\nu}\alpha T\right] = 0,$$

for z = d the displacements u = w = 0.

2. Surfaces z = 0 and z = d are free, and on them $\sigma_{rz} = \sigma_{zz} = 0$.

Boundary conditions in the first case, with account being taken of relations (8) and (9), have the form

$$\frac{d\varphi_n}{dz} - p\Phi_n = 0,\ \frac{d\Phi_n}{dz} + \frac{\nu}{1-\nu}p\varphi_n - \frac{1+\nu}{1-\nu}\alpha T = 0,\ z = 0, \qquad (15)$$
$$\varphi_n = \Phi_n = 0,\ z = d;$$

in the second case

$$\frac{d\varphi_n}{dz} - p\Phi_n = 0,\ \frac{d\Phi_n}{dz} + \frac{\nu}{1-\nu}p\varphi_n - \frac{1+\nu}{1-\nu}\alpha T = 0,\ z = 0,\ d. \qquad (16)$$

Substituting relations (5) and (14) into the boundary conditions, we obtain a system of algebraic equations for A_n, B_n, C_n, D_n in the case of boundary conditions (15):

$$\frac{1}{4(1-\nu)}(A_n + B_n) - p(C_n + D_n) = 0,\ \frac{1}{4(1-\nu)}(B_n - A_n) + p(D_n - C_n) = 2\,\frac{1+\nu}{1-\nu}\,\frac{\alpha p x^2}{(n\pi)^2 + x^2},$$
$$A_n\left[\frac{3-4\nu}{2} + x\right]\exp(-x) + B_n\left[\frac{3-4\nu}{2} - x\right]\exp x + 2(1-\nu)\times$$
$$\times p(C_n\exp(-x) + D_n\exp x) = 0, \qquad (17)$$
$$A_n\left[x - \frac{3-4\nu}{2}\right]\exp(-x) + B_n\left[\frac{3-4\nu}{2} + x\right]\exp x + 2(1-\nu)$$
$$\times p(C_n\exp(-x) - D_n\exp x) = -4(1+\nu)\alpha p\,\frac{(-1)^n x^2}{(n\pi)^2 + x^2},$$

where x = pd. In the case of conditions (16) the first and second equations of system (17) stay the same, but in the third and fourth equations it is necessary to replace $3 - 4\nu/2$ in the square brackets by $-1/2$.

Substituting the coefficents A_n, B_n, C_n, D_n into relation (14), we obtain equations for Φ_n and φ_n. The components of the displacement u and w, depend, in accordance with relations (8) and (9), on φ_n and Φ_n, while the stress tensor components, in accordance with the known Hooke's Law relationships, may be expressed in terms of u and w and their first derivatives. We give the expressions for the stresses σ_{zz} and σ_{rz} for z = d for the case involving clamping of the plate on a rigid base:

$$\sigma_{rz}(r,\ d,\ t) = \frac{I_0 AG}{2c\rho a d^2\mu}\int_0^\infty \exp\left(-\frac{x^2}{4ad^2}\right)J_1\left(x\frac{r}{d}\right) \times \sum_{n=0}^{\infty} F_n(t)\,[A_n\exp(-x) + B_n\exp x]\,dx, \qquad (18)$$
$$\sigma_{zz}(r,\ d,\ t) = \frac{I_0 AG}{2c\rho a d^2\mu}\int_0^\infty \exp\left(-\frac{x^2}{4ad^2}\right)J_0\left(x\frac{r}{d}\right) \times \sum_{n=0}^{\infty} F_n(t)\,[-A_n\exp(-x) + B_n\exp x]\,dx.$$

Relations analogous to relation (18) can also be obtained for the stress tensor components.

Figure 1 shows the dependence of the stress σ_{zz} at the point r = 0, z = d on the parameter α for a copper plate of thickness $d = 2\cdot10^{-3}$ m ($c = 392.9$ J/kg•deg), $\rho = 8.93\cdot10^3$ kg/m³, $\mu = 5\cdot10^3$ m⁻¹, $G = 4\cdot10^4$ MPa, $\alpha = 1.67\cdot10^{-5}$ deg⁻¹) at impulse termination. We assume that the

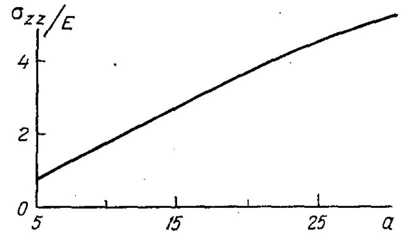

Fig. 1

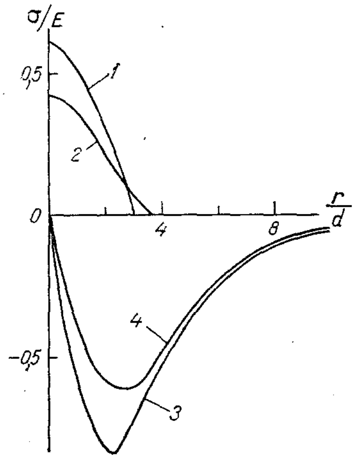

Fig. 2

Fig. 1. Dependence of the stress σ_{zz} on the parameter a. Units for σ_{zz} and a are MPa and cm^{-2}, respectively.

Fig. 2. Dependence of stress σ (curves 1 and 2 are for $\sigma = \sigma_{zz}$; curves 3 and 4 are for $\sigma = \sigma_{rz}$) on distance r/d for various values of a. For curves 1 and 3, $a = 4$ cm^{-2}; for curves 2 and 4, $a = 3$ cm^{-2}. Stress σ is in units of MPa.

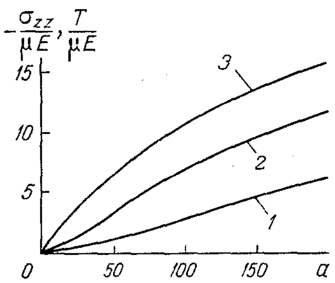

Fig. 3

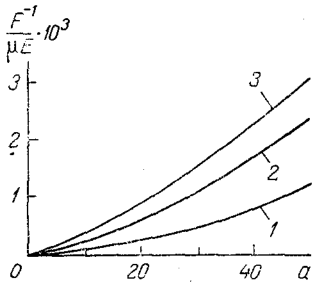

Fig. 4

Fig. 3. Dependence of stress σ_{zz} (curves 1 and 2) and temperature T (curve 3) on a. Values of d for curves 1 and 2 are 0.25 cm and 0.5 cm, respectively. T in °K.

Fig. 4. Dependence of optical intensity F^{-1} on a for various values of d: Curves 1, 2, and 3 are for d values 0.1, 0.25, and 0.5 cm. Units of F^{-1} are in cm^{-1}.

radiative impulse is of rectangular form and of duration $\tau = 10^{-2}$ sec. Then for the energy E absorbed by the plate material we have the relation

$$E = 2\pi I_0 A\tau \int\limits_0^\infty \exp(-ar^2)\, r dr \int\limits_0^d \exp(-\mu z)\, dz = \pi \frac{I_0 A\tau}{a\mu}. \tag{19}$$

It follows from the figure that for more concentrated radiative flows, i.e., with an increase in a, the normal stress σ_{zz} increases almost linearly. Since $\sigma_{zz} > 0$, it is a tensile stress, so that when a certain limiting value of σ_{zz} is reached, depending on the strength of the contact between the plate and the base, the plate may come loose from the base. Figure 2 shows the dependence of the stresses σ_{zz} and σ_{rz}, acting in the plane z = d, on the radial coordinate r. It was found that $r_0 \approx a^{-1/2}$, where r_0 is a characteristic transverse measure of the region in which σ_{zz} differs from zero. The effect of the compressive stresses σ_{rz} in the plane z = d becomes apparent at a much greater distance from the r = 0 axis than in the case of the stresses σ_{zz}.

In the case of a free plate the expression for the stress σ_{zz}, acting in the mean plane, i.e., for z = d/2, has the form

$$\sigma_{zz}\left(r,\ \frac{d}{2},\ t\right) = \frac{EG}{8\pi(1-\nu)c\rho d^2\tau}\int_0^\infty \exp\left(-\frac{x^2}{4ad^2}\right) J_0\left(x\frac{r}{d}\right)\sum_{n=0}^{\infty} F_n(t) \times$$
$$\times\left\{A_n\left[\exp\frac{x}{2} - (1+x)\exp\left(-\frac{x}{2}\right)\right] + B_n\left[\exp\left(\frac{x}{2}\right)(1-x) - \right.\right. \tag{20}$$
$$\left.\left. -\exp\left(-\frac{x}{2}\right)\right] - 4(1+\nu)\frac{\alpha p x^2}{(n\pi)^2+x^2}\left(2\cos\frac{n\pi}{2} - \exp\left(-\frac{x}{2}\right) - \exp\left(\frac{x}{2}\right)\right)\right\} dx.$$

We consider the case in which optical radiation ($A \equiv \mu$) passes through a weakly absorbing glass plate ($\mu d \ll 1$). It is then convenient to refer the values of σ_{zz} to the value μE. Figre 3 shows the values of σ_{zz} at the point $r = 0$, $z = d/2$ at the instant $t = \tau = 1$ sec of impulse termination for various a values ($c\rho = 1.69 \cdot 10^6$ J/m³ deg), $G = 3 \cdot 10^4$ MPa, $\nu = 0.30$, $\alpha = 8.5 \cdot 10^{-6}$ deg⁻¹). Calculations made in accordance with relation (20) show that for $a \lesssim 10$ cm⁻² the values of σ_{zz} differ very little from zero for thin plates ($d \lesssim 0.2$ cm). With an increase in d the stress σ_{zz} increases very sharply for $a \lesssim 10$ cm⁻². It is well known (see [4]) that for calculations of stresses in a plate very broad use is made of a plane stress state approximation in which $\sigma_{zz} \equiv 0$. Relation (20) allows us to estimate the limits of applicability of this approximation. In partiuclar, it was found that its accuracy increases as the parameter a and the plate thickness d decrease. In Fig. 3 we show how the temperature $T(r = 0)$ depends on a.

As the result of nonhomogeneous heating in the plate a thermal lens is formed due to a change in refractive index (volume lens) and to a distortion of the surfaces (surface lens). We limit ourselves here to a determination of the optical density of the surface lens, using expressions (9) and (14) for the displacement of the surface $z = 0$ of the plate:

$$w(r,\ 0,\ t) = \frac{E}{4\pi c\rho d\tau}\int_0^\infty \exp\left(-\frac{x^2}{4ad^2}\right) J_0\left(x\frac{r}{d}\right) F_0(t)\frac{A_0+B_0}{x}\,dx. \tag{21}$$

Since the plate is weakly absorbent, the displacement of the surface $z = d$ is equal in magnitude and opposite in sign to the displacement $w(r, 0, t)$. In calculating the optical intensity F^{-1} of the surface lens for the paraxial rays in integral (21) we express the Bessel function in a series up to the square term. As a result, we obtain

$$F^{-1} = \frac{E(n_0-1)}{2\pi c\rho d^3\tau}\int_0^\infty \exp\left(-\frac{x^2}{4ad^2}\right) xF_0(t)(A_0+B_0)\,dx. \tag{22}$$

Figure 4 shows the dependence of the optical intensity F^{-1} of a glass plate on parameter a. It was found that F^{-1} increases as the parameter a and the thickness d increase.

NOTATION

T, temperature; k, thermal diffusivity; c, specific heat capacity; ρ, density of plate material; r, radial coordinate; z, axial coordinate; t, time; α, temperature coefficient of linear expansion; G, shear modulus; ν, Poisson coefficient; $J_n(x)$, Bessel function of 1st kind; μE, linear density of absorbed energy, J/cm; n_0, index of refraction.

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